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Eigenvalue and Eigenvector Approximate Analysis for Repeated Eigenvalue Problems

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Repeated eigenvalue problems occur in engineering design and analysis practices. The discontinuous nature of these problems has recently attracted significant interest in sensitivity analysis. The objectives of this paper are to present a method for eigenvalue and eigenvector approximate analysis in the presence of repeated eigenvalues and to present an alternate method for the eigenvector sensitivity equations. The method developed for approximate analysis involves a reparamaterization of the multivariable structural eigenvalue problems in terms of a single positive-valued parameter. The resulting equations yield first-order approximations of changes in both the eigenvalues and eigenvectors associated with the repeated eigenvalue problem. Examples are given to demonstrate the application of such equations for sensitivity and approximate analysis.

I. Introduction

THE main objective of this paper is to introduce an eigenvalue/vector first-order approximation technique for the real, symmetric structural eigenproblem for the case of repeated eigenvalues with distinct eigenvalue derivatives. An alternate matrix formulation for the eigenvector derivative will also be presented.

The computation of eigenvector derivatives is hindered by the fact that the eigenvectors are not unique, since any linear combination of eigenvectors corresponding to a repeated eigenvalue is also an eigenvector. Recent publications, however, such as the works of Ojalvo, Mills-Curran,[2] Dailey,[3] Juang et al.,[4] and Bernard,[5] have proposed numerical procedures to reorientate the eigenvectors to formulate a set of differentiable eigenvectors. The work reported in Ref. 4 is distinguished from the rest in that it deals with nonsymmetric matrices and discusses the case when the derivatives of repeated eigenvalues are also repeated. Nevertheless, none of the cited works has yet addressed the issue of eigenvalue/vector approximate analysis in the presence of repeated eigenvalues.

Approximate analysis of eigenvalues/vectors in the presence of repeated eigenvalues is complicated by the apparent discontinuous nature of the eigensolution. This discontinuity produces eigenvalues/vectors that are design variable move dependent; i.e., the ordering of the eigenvalues, and consequently the eigenvectors, are functions of design variable changes. Early works done by Haug et al.[6] Concentrated on deriving sensitivity equations for repeated eigenvalues. Eigenvalues problems studied by those authors had repeated eigenvalues but had distinct first eigenvalue derivatives. It was noted by those authors that repeated eigenvalues of this type are only directionally (Gateaux) differentiable.

This paper will introduce a method to reparameterize the multivariable eigenproblem into an eigenproblem that is in terms of a single positive-valued design parameter. This reparameterization will effectively eliminate the directional dependencies of the original multivariable eigenproblem, allowing the eigenvalues/vectors to be approximated using conventional series expansion techniques.

This paper will also introduce an alternate eigenvector derivative matrix formulation that has an interesting characteristic, namely an elimination of the numerical burden related to repeated factorization of design variable dependent eigensensitivity matrix equations. This formulation eliminates the need for a matrix factorization of the large-order sensitivity equations for every design variable per eigenvalue/vector pair, and instead requires only backsubstitution for the design variable dependent right-hand-side vectors.

II. Derivation of Eigenvalue and Eigenvector Sensitivity Equations

This section will outline the derivations of eigenvalue/vector sensitivity equations for repeated eigenvalue problems with distinct first derivatives. See Kenny[7] for a more detailed derivation of eigenvalue/vector sensitivity equations, as well as eigenvalue/vector approximate analysis. In the following derivation, it is assumed that the original eigenvectors form a nondefective set; i.e., the original set of eigenvectors completely spans an n-dimensional space.

In the case of repeated eigenvalues the eigenvectors are generally nondifferentiable. Therefore, the methods derived for simple (distinct) eigenvalue problems will no longer be valid. This difficulty may be better explained by investigating the differences between simple and repeated eigenvalue problems. The first and most fundamental difference is that the eigenvectors of repeated eigenvalue problems have a great deal of uncertainty compared to those associated with simple eigenvalues. This is due to the fact that any linear combination of them is also a valid eigenvector. The second difference is related to the rank deficiency of the matrix $(K - \lambda M)$, where K and M are the stiffness and the mass matrices, respectively. If repeated eigenvalues occur within a multiplicity of m, then the matrix $(K - \lambda M)$ will be rank deficient by m, whereas it is deficient by one for the simple eigenvalue problem.

To start the derivation, assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the pair of eigenvectors associated with the repeated eigenvalue λ. The differentiable eigenvectors associated with λ may then be represented as a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as

$$\phi_i = Xy_i, \qquad i = 1,2$$

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where X is an $n \times 2$ matrix with x_1 and x_2 as its columns, and $y_1 = \{y_1, y_2\}$ is a constant vector. The differentiable eigenvector should satisfy the n-dimensional eigenvalue equation:

$$(K - \lambda M)\phi_i = 0 \qquad (1)$$

It has been shown in the literature[1-6] that the derivatives of the repeated eigenvalues are the solution of the following reduced eigenvalue problem:

$$\left(\tilde{K} - \frac{\partial \lambda_i}{\partial b}\tilde{M}\right) y_i = 0, \qquad i = 1,2 \qquad (2)$$

where

$$\tilde{K} = X^T\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right) X$$

and

$$\tilde{M} = X^T M X$$

Note that, since x_1 and x_2 form a nondefective set, they can be "forced" to become mass orthonormal by any orthogonalization process (e.g., Gram-Schmidt). Therefore, $\tilde{M}$ can be equal to the identity matrix. Equation (2) provides a means of finding the eigenvalue derivatives as well as the transformation matrix Y with y_1 and y_2, the eigenvectors of Eq. (2), as its columns. The matrix Y will in turn allow the definition of a set of differentiable eigenvectors. If the eigenvalue derivatives are distinct, Eq. (2) yields two simple eigensolutions: $(\partial\lambda_1/\partial b, y_1)$ and $(\partial\lambda_2/\partial b, y_2)$. The differential eigenvectors $\phi_1 = Xy_1$ and $\phi_2 = Xy_2$ correspond to the eigenvalue derivatives $\partial\lambda_1/\partial b$ and $\partial\lambda_2/\partial b$, respectively.

To find the eigenvector derivative $\partial\phi_i/\partial b$ for the situation $\lambda_1 = \lambda_2 \equiv \lambda$, one can start with the equation for the eigenvector derivative

$$(K - \lambda M)\frac{\partial \phi_i}{\partial b} = \frac{\partial \lambda_i}{\partial b} M\phi_i - \left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i$$

$$\equiv f_i, \qquad i = 1,2 \qquad (3)$$

Many methods for finding the solution of Eq. (3) are available in the literature.[1-4] However, a procedure that is different from those methods is presented here.

This procedure suggests that the eigenvector derivative can be rewritten as a sum of an arbitrary vector u_i and the products of two undetermined constants and the differentiable eigenvectors, i.e.,

$$\frac{\partial \phi_i}{\partial b} = u_i + c_{i1}\phi_1 + c_{i2}\phi_2 \qquad i = 1,2 \qquad (4)$$

where u_i is required to be mass orthogonal with respect to the complementary solution $(c_{i1}\phi_1 + c_{i2}\phi_2)$, i.e.,

$$u_i^T M(c_{i1}\phi_1 + c_{i2}\phi_2) = 0, \qquad i = 1,2$$

which can be restated as

$$u_i^T M\phi_1 = 0, \qquad i = 1,2 \qquad (5)$$

$$u_i^T M\phi_2 = 0 \qquad i = 1,2 \qquad (6)$$

With the aid of Eqs. (2) and (4) and the identity

$$\frac{\partial \lambda_i}{\partial b} = \phi_i^T\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i,$$

Eq. (3) can be written as

$$(K - \lambda M)u_i = \left[\phi_i^T\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i\right](M\phi_i)$$

$$- \left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i, \qquad i = 1,2 \qquad (7)$$

whose solution u_i, has to satisfy the linear equality constraints of Eqs. (5) and (6). According to the theorem of Lagrange multipliers,[8] a unique solution of Eq. (7) subject to the linear constraints Eqs. (5) and (6) can be found as the solution of the following equation:

$$(K - \lambda M)u_i + \mu_{i1}M\phi_1 + \mu_{i2}M\phi_2$$

$$= -\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i, \qquad i = 1,2 \qquad (8)$$

Nevertheless, since Eq. (8) is a function of the differential eigenvectors ϕ_1 and ϕ_2, which are different for each design variable, it has to be factored and solved for every design variable. Therefore, solving Eq. (8) in a design optimization environment becomes very cumbersome. Consequently, deriving an alternate equation for the eigenvector derivatives becomes desirable.

Let u_i be redefined as a linear combination of v_i, $i = 1,2$, where $u_i = Vy_i$ and V is an $n \times 2$ matrix with v_i as its columns. With this new definition and the definition of ϕ_i, Eqs. (5–7) may be reexpressed as

$$y_i^T V^T MXy_1 = 0 \qquad i = 1,2 \qquad (9)$$

$$y_i^T V^T MXy_2 = 0, \qquad i = 1,2 \qquad (10)$$

and

$$(K - \lambda M)Vy_i = \left[\phi_i^T\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i\right](MXy_i)$$

$$- \left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)(Xy_i), \qquad i = 1,2 \qquad (11)$$

Since the eigenvectors y_1 and y_2 are linearly independent, it can be proved that Eqs. (9–11) are identical to

$$v_i^T Mx_1 = 0 \qquad i = 1,2 \qquad (12)$$

$$v_i^T Mx_2 = 0 \qquad i = 1,2 \qquad (13)$$

and

$$(K - \lambda M)v_i = \left[\phi_i^T\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)\phi_i\right](Mx_i)$$

$$- \left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right)x_i, \qquad i = 1,2 \qquad (14)$$

Finally, with the aid of the theorem of Lagrange multipliers, the solution of Eq. (14) subjected to constraints of Eqs. (12) and (13) is equal to the solution of the following equation:

$$\begin{bmatrix} K - \lambda M & Mx_1 & Mx_2 \\ x_1^T M & 0 & 0 \\ x_2^T M & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_i \\ \mu_{i1} \\ \mu_{i2} \end{Bmatrix}$$

$$= \begin{Bmatrix} -\left(\frac{\partial K}{\partial b} - \lambda \frac{\partial M}{\partial b}\right) x_i \\ 0 \\ 0 \end{Bmatrix}, \qquad i = 1,2 \qquad (15)$$

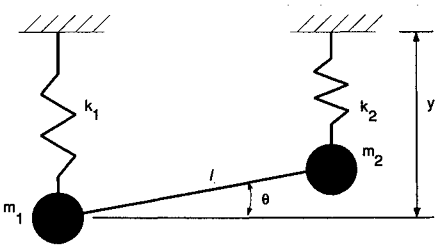

Fig. 1 Two-degrees-of-freedom spring-mass system.

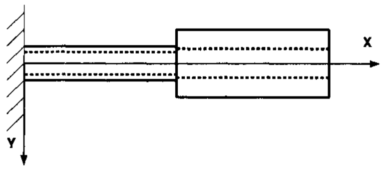

Fig. 2 Stepped cantilever beam.

Note that the left-hand side is independent of design variables and Eq. (15) can be solved without the differential eigenvectors being known in advance. The eigenvector derivative is now defined as

$$\frac{\partial \phi_i}{\partial b} = u_i + c_{i1}\phi_1 + c_{i2}\phi_2, \qquad i = 1,2 \quad (16)$$

with the understanding that $u_i = Vy_i$.

To continue the derivation of $\partial\phi_i/\partial b$, equations for the constants c_{i1} and c_{i2} in Eq. (4) or (16) need to be derived. This can be partially done by using the first two normalization conditions:

$$\phi_i^T M \phi_i = 1, \qquad i = 1,2$$

whose design derivatives yield

$$c_{11} = -\frac{1}{2}\phi_1^T \frac{\partial M}{\partial b}\phi_1 \quad (17)$$

and

$$c_{22} = -\frac{1}{2}\phi_2^T \frac{\partial M}{\partial b}\phi_2 \quad (18)$$

Regarding constants c_{12} and c_{21}, equations similar to those given in Refs. 2 and 3 can be derived by taking the second derivative of Eq. (1) to obtain the following equalities:

$$c_{12} = \left\{\phi_2^T\left(\frac{\partial^2 K}{\partial b^2} - 2\frac{\partial \lambda_1}{\partial b}\frac{\partial M}{\partial b} - \lambda\frac{\partial^2 M}{\partial b^2}\right)\phi_1 + 2\phi_2^T\left(\frac{\partial K}{\partial b} - \lambda\frac{\partial M}{\partial b}\right)u_1\right\}\bigg/ 2\left(\frac{\partial \lambda_1}{\partial b} - \frac{\partial \lambda_2}{\partial b}\right) \quad (19)$$

and

$$c_{21} = \left\{\phi_1^T\left(\frac{\partial^2 K}{\partial b^2} - 2\frac{\partial \lambda_2}{\partial b}\frac{\partial M}{\partial b} - \lambda\frac{\partial^2 M}{\partial b^2}\right)\phi_2 + 2\phi_1^T\left(\frac{\partial K}{\partial b} - \lambda\frac{\partial M}{\partial b}\right)u_2\right\}\bigg/ 2\left(\frac{\partial \lambda_2}{\partial b} - \frac{\partial \lambda_1}{\partial b}\right) \quad (20)$$

It is worthwhile to mention here that the derivative of the orthogonal condition between ϕ_1 and ϕ_2, $\phi_1^T M\phi_2 = 0$, should also be satisfied by the eigenvector derivatives $\partial\phi_1/\partial b$ and $\partial\phi_2/\partial b$. That is, the eigenvector derivatives should preserve the following identity:

$$\frac{\partial \phi_1^T}{\partial b} M\phi_2 + \phi_1^T M \frac{\partial \phi_2}{\partial b} + \phi_1^T\frac{\partial M}{\partial b}\phi_2 = 0$$

that results in an equality with c_{12} and c_{21} as

$$c_{12} + c_{21} + \phi_1^T \frac{\partial M}{\partial b}\phi_2 = 0 \quad (21)$$

The relation of Eq. (21) can also be proved by noting that

$$u_2^T\left(\frac{\partial K}{\partial b} - \lambda\frac{\partial M}{\partial b}\right)\phi_1 = u_1^T\left(\frac{\partial K}{\partial b} - \lambda\frac{\partial M}{\partial b}\right)\phi_2$$

which is obtained by alternately premultiplying Eq. (8) by u_2^T with $i = 1$, and u_1^T with $i = 2$. Thus, once c_{12} is found by using Eq. (19), c_{21} can be simply obtained via Eq. (21), which is computationally more efficient than using Eq. 20.

It should be noted that, although eigenvalue/vector derivatives associated with repeated eigenvalues can be determined

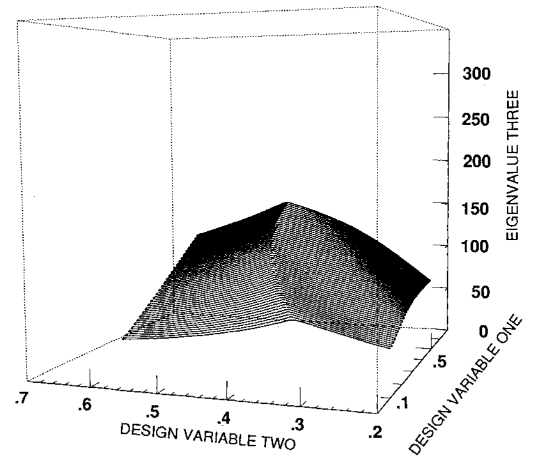

Fig. 3 Variations of eigenvalue 3 vs design variables one and two.

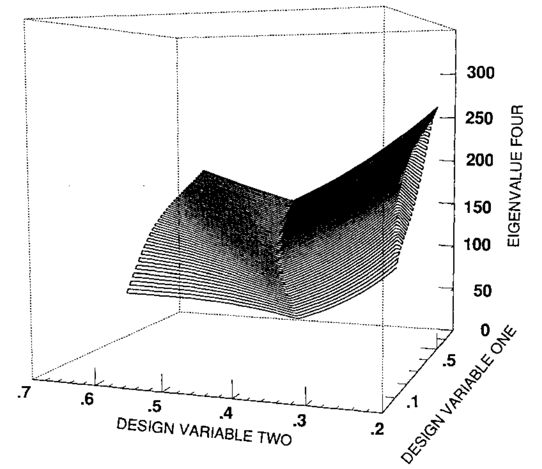

Fig. 4 Variations of eigenvalue 4 vs design variables one and two.

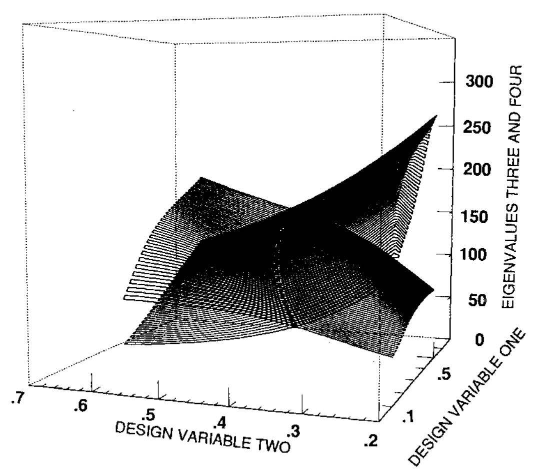

Fig. 5 Eigenvalue variations vs design variables one and two.

by Eq. (2) and Eqs. (15–21), they cannot be directly used to estimate the change in the eigensolution. This will be subject of discussion in Sec. III. Examples will be presented later to demonstrate the application of the sensitivity equations of Eqs. (15–21).

III. Eigenvalue and Eigenvector Approximate Analysis

Most of the computational cost associated with any analysis or optimization procedure requiring iterative eigensolutions is related to the eigensolutions themselves. This is true for even relatively small problems (less than 100 degrees of freedom). The undeniable fact is that eigensolutions are extremely time consuming and computationally burdensome. To alleviate some of this burden, the development of methods to efficiently approximate eigensolutions becomes desirable. One such method is the Taylor's series expansion, in which the new eigenvalues/vectors are linearly approximated as

$$\lambda_i(\boldsymbol{b}^*) \approx \lambda_i(\boldsymbol{b}^0) + \sum_{j=1}^{ndev} \frac{\partial \lambda_i}{\partial b_j} \Delta b_j \tag{22}$$

$$\boldsymbol{x}_i(\boldsymbol{b}^*) \approx \boldsymbol{x}_i(\boldsymbol{b}^0) + \sum_{j=1}^{ndev} \frac{\partial \boldsymbol{x}_i}{\partial b_j} \Delta b_j \tag{23}$$

where $\boldsymbol{b}^*$ and $\boldsymbol{b}^0$ are the new and the current design variables, respectively, and Δb_j is the change in design variable b_j. Note that the eigenvalue/vector derivatives appearing in Eqs. (22) and (23) should be associated with the eigenvalues ordered according to their magnitudes. Difficulties arise when Eqs. (22) and (23) are applied to the repeated eigenvalue problem because the subeigenvalue problem [Eq. (2)] fails to indicate the correspondence between the eigenvalue derivatives and the distinct eigenvalues after the design is changed. A different set of equations, as suggested by Refs. 1 and 6 should be derived to account for the direction of change in $\boldsymbol{b}$.

Let a repeated eigenvalue problem formulated at the current design $\boldsymbol{b}^0$, be represented as

$$K(\boldsymbol{b}^0)\boldsymbol{x}^0 = \lambda^0 M(\boldsymbol{b}^0)\boldsymbol{x}^0$$

where the superscript "0" indicates the quantity defined at $\boldsymbol{b}^0$. Next, introducing a design change, $\Delta \boldsymbol{b}$, the new design variable $\boldsymbol{b}^*$, is then given as

$$\boldsymbol{b}^* = \boldsymbol{b}^0 + \Delta \boldsymbol{b}$$

which yields an eigenvalue problem as

$$K(\boldsymbol{b}^*)\boldsymbol{x}^* = \lambda^* M(\boldsymbol{b}^*)\boldsymbol{x}^*$$

where the superscript asterisk denotes the quantity defined at the new design.

Introducing an intermediate design variable, $\boldsymbol{b}(\epsilon) = \boldsymbol{b}^0 + \epsilon\Delta\boldsymbol{b}$, the preceding two eigenvalue equations can be collectively represented by a single equation:

$$K(\boldsymbol{b}(\epsilon))\boldsymbol{x}(\epsilon) = \lambda(\epsilon) M(\boldsymbol{b}(\epsilon))\boldsymbol{x}(\epsilon) \tag{24}$$

where ϵ, a real parameter, ranges from 0 to 1. Note that the current and the new eigenvalue equations can be realized by substituting the value of ϵ by 0 and 1, respectively. Letting ϵ, which is always positive, be assigned as the only design variable, the sensitivity equation derived in the previous section can be applied here to find the eigenvalue/vector derivatives of Eq. (24) with respect to ϵ at $\epsilon = 0$. Note that Eq. (24) entertains a pair of repeated eigenvalues at $\epsilon = 0$.

The eigenvalue derivatives of Eq. (24) at $\epsilon = 0$ are the eigenvalues of the following subeigenvalue problem:

$$(\tilde{K} - \gamma_i \tilde{M})\boldsymbol{y}_i = 0 \tag{25}$$

where $\tilde{K} = X^T(K' - \lambda M')X$ and $\tilde{M} = I$, the identity matrix.

The detailed representations of K' and M' are given as

$$K' = \left. \frac{\mathrm{d}K(\boldsymbol{b}^0 + \epsilon\Delta\boldsymbol{b})}{\mathrm{d}\epsilon} \right|_{\epsilon=0} = \frac{\partial K}{\partial \boldsymbol{b}} \Delta \boldsymbol{b} = \sum_{i=1}^{ndev} \frac{\partial K}{\partial b_i} \Delta b_i \tag{26}$$

and similarly,

$$M' = \sum_{i=1}^{ndev} \frac{\partial M}{\partial b_i} \Delta b_i \tag{27}$$

The eigenvector derivative ϕ_i', corresponding to the eigenvalue derivative γ_i, is now defined as

$$\phi_i' = V\boldsymbol{y}_i + c_{i1}\phi_1 + c_{i2}\phi_2, \qquad i = 1,2 \tag{28}$$

where $\boldsymbol{y}_i$ is the eigenvector of Eq. (25), and $\boldsymbol{v}_i$ are the solutions of the following equation:

$$\begin{bmatrix} K - \lambda M & M\boldsymbol{x}_1 & M\boldsymbol{x}_2 \\ \boldsymbol{x}_1^T M & 0 & 0 \\ \boldsymbol{x}_2^T M & 0 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{v}_i \\ \mu_{i1} \\ \mu_{i2} \end{Bmatrix} = \begin{Bmatrix} -(K' - \lambda M')\boldsymbol{x}_i \\ 0 \\ 0 \end{Bmatrix}, \qquad i = 1,2 \tag{29}$$

Now that Eq. (29) is very similar to Eq. (15) in which $\partial K/\partial b$ and $\partial M/\partial b$ are substituted by K' and M' defined by Eqs. (26) and (27). Furthermore, the constants c_{i1} and c_{i2}, $i = 1,2$ in Eq. (28) can be obtained by using Eqs. (17), (18), and any two of the three equations, Eqs. (19) or (20) and (21), in which $\partial K/\partial b$ and $\partial M/\partial b$ should be replaced by K' and M' and $\partial^2 K/\partial b^2$ and $\partial^2 M/\partial b^2$ should be replaced by K'' and M''. K'' and M'' are defined as

$$K'' = \left. \frac{\mathrm{d}^2 K(\boldsymbol{b}^0 + \epsilon\Delta\boldsymbol{b})}{\mathrm{d}^2\epsilon} \right|_{\epsilon=0} = \sum_{i=1}^{ndev} \sum_{j=1}^{ndev} \frac{\partial^2 K}{\partial b_i \partial b_j} \Delta b_i \Delta b_j$$

and

$$M'' = \sum_{i=1}^{ndev} \sum_{j=1}^{ndev} \frac{\partial^2 M}{\partial b_i \partial b_j} \Delta b_i \Delta b_j$$

The first-order approximation of the new eigenvalue, at $\epsilon = 1$, can then be obtained by using the Taylor's series expansion with respect to ϵ about $\epsilon = 0$ as

$$i = 1,2 \quad \lambda_i^* \approx \lambda_i^0 + \gamma_i, \tag{30}$$

whereas γ_1 is less than γ_2. Therefore, as a result, λ_1^* is always less than λ_2^*. Similarly, the first-order approximation of eigenvectors can be obtained by

$$i = 1,2 \quad \phi_i^* \approx \phi_i^0 + \phi_i', \tag{31}$$

where ϕ_i^0 is the differentiable eigenvector at $\epsilon = 0$ determined by $\boldsymbol{y}_i$, the eigenvector corresponding to γ_i in Eq. (25).

IV. Examples

A simple two-degrees-of-freedom spring-mass system[1] as shown in Fig. 1 is presented here as an example to demonstrate the application of the sensitivity equations derived previously.

Letting $x_1 = y$ and $x_2 = L\theta$, the associated eigenvalue problem is

$$\left(\begin{bmatrix}(k_1+k_2) & -k_2\\ -k_2 & k_2\end{bmatrix} - \lambda\begin{bmatrix}(m_1+m_2) & -m_2\\ -m_2 & m_2\end{bmatrix}\right)\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix} \tag{32}$$

The characteristic equation of the problem is

$$\lambda^2 - \lambda\left(\frac{k_1}{m_1}+\frac{k_2}{m_2}\right) + \frac{k_1k_2}{m_1m_2} = 0$$

The preceding equation can be treated as a simple eigenvalue problem and can be solved to provide the following solution:

$$\lambda_1 = \frac{k_1}{m_1}, \qquad x_1 = (1,1)^T/\sqrt{m_1} \tag{33}$$

$$\lambda_2 = \frac{k_2}{m_2}, \qquad x_2 = (0,-1)^T/\sqrt{m_2} \tag{34}$$

Letting $k_2 = \alpha k_1$ and $m_2 = \beta m_1$, various types of eigenvalue problems can be observed depending on the particular choice of α and β. For example, if $\alpha = \beta$, Eq. (32) yields a repeated eigenvalue problem. Note that in this case the rank of $K - \lambda M$ in Eq. (32) becomes zero for $\lambda = k_1/m_1$. Therefore, any pair of linearly independent vectors can be the eigenvectors. After applying the normalizing conditions, one can select a set of eigenvectors as

$$x_1 = \frac{1}{\sqrt{(1+\beta)m_1}}\begin{Bmatrix}1\\0\end{Bmatrix} \tag{35}$$

$$x_2 = \frac{1}{\sqrt{m_1}}\begin{Bmatrix}\sqrt{\beta}/\sqrt{1+\beta}\\ \sqrt{1+\beta}/\sqrt{\beta}\end{Bmatrix} \tag{36}$$

Let α be the only design variable and β remain as a constant. The exact eigensensitivity evaluated at the design, obtained where $\alpha = \beta$ by differentiating Eqs. (33) and (34), is listed as follows

$$\frac{d\lambda_1}{d\alpha} = 0 \qquad \frac{dx_1}{d\alpha} = 0$$

$$\frac{d\lambda_2}{d\alpha} = \frac{k_1}{\beta m_1} \qquad \frac{dx_2}{d\alpha} = 0$$

According to Eqs. (35) and (36), the matrix X is

$$X = \frac{1}{\sqrt{(1+\beta)m_1}}\begin{bmatrix}1 & \sqrt{\beta}\\ 0 & \dfrac{1+\beta}{\sqrt{\beta}}\end{bmatrix}$$

and

$$\left(\frac{\partial K}{\partial \alpha} - \lambda\frac{\partial M}{\partial \alpha}\right) = k_1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Furthermore,

$$X^T\left(\frac{\partial K}{\partial \alpha} - \lambda\frac{\partial M}{\partial \alpha}\right)X = \frac{k_1}{(1+\beta)m_1}\begin{bmatrix}1 & \dfrac{-1}{\sqrt{\beta}}\\ \dfrac{-1}{\sqrt{\beta}} & \dfrac{1}{\beta}\end{bmatrix}$$

Table 1 Perturbation history of eigenvector 3 (lateral displacement)

Y DOF at node	x	$x_{\beta_1}^*$	$x_{\beta_2}^*$
1	0	0	0
5	37.190	37.183	0
9	20.007	20.006	0
13	− 13.803	− 13.804	0
17	− 7.861	− 7.862	0
21	25.418	25.420	0

Table 2 Perturbation history of eigenvector 4 (axial displacement)

X DOF at node	x	$x_{\beta_1}^*$	$x_{\beta_2}^*$
1	0	0	0
5	7.5961	7.5960	0
9	14.814	14.813	0
13	19.323	19.322	0
17	20.899	20.899	0
21	21.433	21.433	0

To simplify the computation, set $\alpha = \beta = 1$. The solution of the subeigenvalue problem [Eq. (2)] gives $\partial\lambda_1/\partial\alpha = 0$ and $\partial\lambda_2/\partial\alpha = k_1/m_1$, with $y_1 = (1/\sqrt{2}, 1/\sqrt{2})^T$ and $y_2 = (1/\sqrt{2}, -1/\sqrt{2})^T$. The differentiable eigenvectors corresponding to $\partial\lambda_1/\partial\alpha = 0$ and $\partial\lambda_2/\partial\alpha = k_1/m_1$ are then obtained, respectively, as

$$\phi_1 = Xy_1 = (1,1)^T/\sqrt{m_1}$$

and

$$\phi_2 = Xy_2 = (0,-1)^T\sqrt{m_1}.$$

It is interesting to note that the differential eigenvector so obtained are identical to the simple eigenvectors presented by Eqs. (33) and (34).

Equations (2), (15–19), and (21) are needed to determine the eigenvector derivatives. For the first differentiable eigenvector, one obtains $\nu_1 = 0$, $\mu_{11} = \mu_{12} = 0$. Moreover, $c_{11} = 0$ as $\partial M/\partial\alpha = 0$ and $c_{12} = 0$ as $\partial^2 K/\partial\alpha^2 = 0$ and $\nu_1 = 0$. Consequently, $\partial x_1/\partial\alpha = 0$. Regarding the second differentiable eigenvector, one obtains $\nu_2 = 0$, $\mu_{21} = 0$, and $\mu_{22} = -k_1/m_1$, and c_{22} and c_{21} are zero. As a result, $\partial x_2/\partial\alpha = 0$.

To study the approximate analysis of the repeated eigenvalue problem, consider both α and β as design variables. Letting the current design be specified as $\alpha^0 = \beta^0 = 1$, one has a pair of repeated eigenvalues

$$\lambda_1^0 = \lambda_2^0 = k_1/m_1$$

with corresponding eigenvectors obtained by substituting 1 for β in Eqs. (35) and (36):

$$x_1^0 = (1,0)^T/\sqrt{2m_1}$$

$$x_2^0 = (1,2)^T/\sqrt{2m_1}$$

As for the perturbed design, one has $\alpha^* = 1 + \Delta\alpha$ and $\beta^* = 1 + \Delta\beta$ with an assumption that $\Delta\alpha > \Delta\beta$. In this case the perturbed eigensolution is

$$\lambda_1^* = k_1/m_1, \qquad \phi_1^* = (1,1)^T/\sqrt{m_1} \tag{37}$$

$$\lambda_2^* = (1+\Delta\alpha)k_1/[(1+\Delta\beta)m_1]$$
$$\phi_2^* = (0,-1)^T/\sqrt{(1+\Delta\beta)m_1} \tag{38}$$

The eigenvalue problem [Eq. (32)] may be reparameterized in terms of the intermediate design variable ϵ:

$$\begin{Bmatrix}\alpha(\epsilon)\\ \beta(\epsilon)\end{Bmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix} + \epsilon\begin{Bmatrix}\Delta\alpha\\ \Delta\beta\end{Bmatrix}$$

Table 3a Approximate analysis of eigenvector 3 for perturbation 1

X DOF at node	ϕ^0	ϕ'	ϕ^*_{Exact}	ϕ^*_{Approx}
2	1.9140	− 0.0972e − 2	1.9128	1.9130
6	9.4505	− 0.4420e − 2	9.4452	9.4461
10	16.5160	− 0.6138e − 2	16.5080	16.5099
14	19.8110	− 1.2712e − 2	19.7970	19.7983
18	21.1320	− 1.8087e − 2	21.1120	21.1139

Table 3b Approximate analysis of eigenvector 4 for perturbation 1

Y DOF at node	ϕ^0	ϕ'	ϕ^*_{Exact}	ϕ^*_{Approx}
2	4.8653	1.6680e − 2	4.8789	4.8820
6	41.7220	0.1165	41.8120	41.8385
10	7.0410	− 2.7260e − 2	7.0083	7.0137
14	− 16.0760	2.8072e − 2	− 16.0380	− 16.0479
18	− 0.8723	1.1553e − 2	− 0.8602	− 0.8607

which results in a set of K and M matrices as

$$K = k_1\begin{bmatrix} 1+\alpha(\epsilon) & -\alpha(\epsilon) \\ -\alpha(\epsilon) & \alpha(\epsilon) \end{bmatrix}$$

$$M = m_1\begin{bmatrix} 1+\beta(\epsilon) & -\beta(\epsilon) \\ -\beta(\epsilon) & \beta(\epsilon) \end{bmatrix}$$

Therefore, the derivatives of K and M with respect to ϵ are given as

$$K' = \Delta\alpha k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M' = \Delta\beta m_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Furthermore, $K'' = M'' = 0$.

The matrix $\tilde{K}$ in Eq. (25) is then given as

$$\tilde{K} = X^T(K' - \lambda M')X = \frac{(\Delta\alpha - \Delta\beta)k_1}{2m_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where the matrix X is given as

$$X = \frac{1}{\sqrt{2m_1}}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Therefore, the eigensolution of Eq. (25) is given as

$$\gamma_1 = 0, \qquad y_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\gamma_2 = (\Delta\alpha - \Delta\beta)k_1/m_1, \qquad y_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$$

where $\gamma_2 > \gamma_1$ because $\Delta\alpha > \Delta\beta$, as previously assumed. As a result, the differentiable eigenvectors are obtained as

$$\phi_1^0 = (1,1)^T/\sqrt{m_1}, \qquad \phi_2^0 = (0,1)^T/\sqrt{m_1}$$

The corresponding eigenvector derivatives [Eq. (28)] respective to ϵ can be readily obtained by solving Eq. (29). In this example it can be shown that $v_1 = v_2 = (0,0)^T$. Furthermore, the constants in Eq. (28) are obtained with the values $c_{11} = c_{12} = c_{21} = 0$ and $c_{22} = -\Delta\beta/2$. Therefore, the eigenvector derivatives are given as

$$\phi_1' = (0,0)^T, \qquad \phi_2' = \frac{-\Delta\beta}{2\sqrt{m_1}}(0, -1)^T$$

Finally, the first-order approximation of the smallest eigenvalue and its associated eigenvector at the new design are obtained following Eqs. (30) and (31) as

$$\lambda_1^* \approx \lambda_1^0 + \gamma_1 = k_1/m_1$$

$$\phi_1^* \approx \phi_1^0 + \phi_1' = (1,1)^T/\sqrt{m_1}$$

The second eigenvalue and eigenvector pair can also be approximated with the following equations:

$$\lambda_2^* \approx \lambda_2^0 + \gamma_2 = \frac{k_1}{m_1}(1 + \Delta\alpha - \Delta\beta)$$

$$\phi_2^* \approx \phi_2^0 + \phi_2' = \frac{1}{\sqrt{m_1}}\left(1 - \frac{\Delta\beta}{2}\right)(0, -1)^T$$

It is a simple matter to prove that the preceding eigenvalues and eigenvectors are indeed the first-order approximations of the exact ones given in Eqs. (37) and (38) with the assumption that the perturbations $\Delta\alpha$ and $\Delta\beta$ are small.

V. Numerical Validation

The goal of this section is to present a simple finite element model that has characteristics which are well understood and to use it to validate the formulation for eigenvalue/vector approximate analysis in the presence of repeated eigenvalues. It should be mentioned that a priori knowledge regarding the characteristics of repeated eigenvalues is rarely available. This is particularly true when complex models with multiple design variables are considered. With more than two design variables, the transformation from repeated to distinct eigenvalues becomes difficult, if not impossible, to visualize. Fortunately, a priori knowledge is not required for the success of the methods developed in Secs. II and III. In fact, the opposite is true. Essentially all that is required is knowledge of the particular design variable changes.

The first part of this validation example will be used to introduce the characteristics of the particular model being considered. This will be done by carefully monitoring the design variable perturbations and examining the resulting change in the eigensolutions. To facilitate this, only one design variable at a time is allowed to be varied.

The model used in this study is a stepped cantilever beam shown in Fig. 2. The finite element model consists of 21 nodes,

Table 4a Approximate analysis of eigenvector 3 for perturbation 2

Y DOF at node	ϕ^0	ϕ'	ϕ^*_{Exact}	ϕ^*_{Approx}
2	4.8653	1.6684e − 2	4.8517	4.8486
6	41.7220	− 0.1165	41.6320	41.6055
10	7.0410	2.7260e − 2	7.0726	7.0683
14	− 16.0760	− 2.8072e − 2	− 16.1150	− 16.1041
18	− 0.8723	− 1.1553e − 2	− 0.8844	− 0.8838

Table 4b Approximate analysis of eigenvector 4 for perturbation 2

X DOF at node	ϕ^0	ϕ'	ϕ^*_{Exact}	ϕ^*_{Approx}
2	1.9140	0.9725e − 3	1.9151	1.9150
6	9.4505	0.4420e − 2	9.4559	9.4549
10	16.5160	0.6137e − 2	16.5240	16.5221
14	19.8110	1.2712e − 2	19.8260	19.8237
18	21.1320	1.8087e − 2	21.1520	21.1501

each with 3 degrees of freedom (2 translational and 1 rotational). The model has been divided into twenty tubular beam elements. The ratio between the inner and the outer radii is fixed in this study. The elements have been subdivided into groups of design variables. The outer radius of the first ten elements is considered as design variable one (b_1) and the outer radius of the remaining elements is design variable two (b_2).

At the nominal design, $b_1 = 0.22699$ and $b_2 = 0.36259$, the frequency of the third bending mode is equal to the first longitudinal mode, which corresponds to the third and fourth eigenvalues, respectively. In this study eigenvalues are considered repeated if their percent difference is $< 1.87\text{E-}5\%$.

Figures 3 and 4 display individually how eigenvalues three and four vary with respect to progressive changes in design variables one and two. The combined eigenvalue variations are shown in Fig. 5. These figures clearly show that the slopes (i.e., eigenvalue derivatives) are discontinuous across the "ridges" where the eigenvalues are repeated.

Tables 1 and 2 show the perturbation histories of eigenvectors three and four for a selected number of degrees of freedom. In these tables the columns labeled $x^*_{b_i}$ are the eigenvectors evaluated when design variable b_i is perturbed by a positive Δb_i. These tables explicitly show the reorientation that must be performed to obtain a set of differentiable eigenvectors. To better understand this reordering, one can investigate the subeigenvalue problem [Eq. (2)]. Assuming that design variable one will be perturbed in a positive sense, Eq. (2) yields an eigensolution of

$$\frac{\partial\lambda_3}{\partial b_1} = 195191.2, \qquad y_1 = (1,0)^T$$

and

$$\frac{\partial\lambda_4}{\partial b_1} = 378836.8, \qquad y_2 = (0,1)^T$$

Similarly, for a positive Δb_2,

$$\frac{\partial\lambda_3}{\partial b_2} = -237167.5, \qquad y_1 = (0,1)^T$$

and

$$\frac{\partial\lambda_4}{\partial b_2} = 149976.9, \qquad y_2 = (1,0)^T$$

It can be seen that, with a positive Δb_1, with $\Delta b_2 = 0$, the corresponding differentiable eigenvectors are $\phi_3 = Xy_1 = x_3$ and $\phi_4 = Xy_2 = x_4$. Thus, for this type of design change, switching of modes is not observed. As indicated in column three of Tables 1 and 2, the newly perturbed eigenvectors three and four remain as bending and longitudinal eigenvectors that have nonzero lateral and axial displacement components, respectively. However, when a positive Δb_2, with $\Delta b_1 = 0$, is introduced, the corresponding differentiable eigenvectors are $\phi_3 = Xy_1 = x_4$ and $\phi_4 = Xy_2 = x_3$. This transformation states that the longitudinal mode becomes the bending mode, and the bending mode becomes the longitudinal mode (i.e., an exact switch has taken place). As indicated in column four of Table 1, the newly perturbed eigenvector three now becomes a longitudinal mode that shows no lateral displacement. The same can be observed in column four of Table 2, in which the newly perturbed eigenvector four is switched to a bending mode (i.e., zero axial displacement).

It should be noted that these unitary diagonal matrices, $Y = [y_1, y_2]$, are special cases of the more general situation that occurs when the eigenvectors are expressed as linear combinations of the pure bending and longitudinal modes. In the preceding discussion design variable perturbations have been limited to single variable positive perturbations. A similar analysis can be performed using negative perturbations. The results will be exactly opposite: A negative perturbation in design variable one will cause the two modes to switch, whereas a negative perturbation in design variable two will not cause the modes to switch.

The characteristics of this problem can be described by examining the ratio of the design variables. When the eigenvalues are equal, b_2/b_1 has a particular value, C_r, and when changes in either b_2 or b_1 increase C_r, the original modes will switch. The opposite is also true. When changes in b_2 or b_1 decrease C_r, the modes will not switch.

Numerical validation of the method presented in Sec. III will be accomplished by investigating two types of design variable perturbations. The first will be a perturbation of both design variables in such a way as to increase C_r, thereby causing the original bending and longitudinal modes to switch. The second perturbation will also permit both design variables to vary simultaneously; however, the perturbation will be such that the ratio C_r decreases. It will be shown that, regardless of the design change, the methods developed in this paper can be used to accurately predict first-order changes in both the eigenvalues and eigenvectors associated with repeated eigenvalues.

Results are presented in the following format. First, the design variable perturbation is given, followed by the transformation required to determine the set of differentiable eigenvectors. Next, first-order approximations of the eigenvalues are presented. Furthermore, Tables 3a, b and 4a, b contain the differentiable eigenvectors as well as their first-order approximations and their exact values for perturbation 1 and perturbation 2, respectively.

Perturbation 1:

$$\Delta b = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} = \begin{pmatrix} -2.2699\text{e}-4 \\ 3.6259\text{e}-4 \end{pmatrix}$$

Transformation matrix:

$$Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalue 3:

$$\lambda_3^* = \lambda^0 + \lambda_3' = 53488.06 + (-171.992)$$

$$= 53316.07 \text{ (Approx)}$$

$$\lambda_3^* = 53316.28 \text{ (Exact)}$$

Eigenvalue 4:

$$\lambda_4^* = \lambda^0 + \lambda_4' = 53488.06 + 10.073$$

$$= 53498.13 \text{ (Approx)}$$

$$\lambda_4^* = 53497.89 \text{ (Exact)}$$

Perturbation 2:

$$\Delta b = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} = \begin{pmatrix} 2.2699\text{e}-4 \\ -3.6259\text{e}-4 \end{pmatrix}$$

Transformation matrix

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalue 3:

$$\lambda_3^* = \lambda^0 + \lambda_3' = 53488.06 + (-10.073)$$

$$= 53477.99 \text{ (Approx)}$$

$$\lambda_3^* = 53477.75 \text{ (Exact)}$$

Eigenvalue 4:

$$\lambda_4^* = \lambda^0 + \lambda_4' = 53488.06 + 171.992$$

$$= 53660.05 \text{ (Approx)}$$

$$\lambda_4^* = 53660.27 \text{ (Exact)}$$

VI. Concluding Remarks

In this paper a method is presented and applied to the real symmetric structural eigenproblem for eigenvalue/vector approximate analysis in the presence of repeated eigenvalues with distinct first eigenvalue derivatives.

The idea of a differentiable eigenvector is first introduced for the case of repeated eigenvalues with distinct first eigenvalue derivatives. A set of differentiable eigenvectors is obtained via a linear transformation of the original eigenvectors. The transformation matrix and the eigenvalue derivatives are obtained by solving a subeigenvalue problem. The additional conditions required to uniquely define the eigenvector derivative are obtained by taking the second derivative of the structural eigenvalue problem.

An eigenvalue/vector approximate analysis is then suggested in the paper in which the perturbed eigenvalue problem is continuously transformed to an eigenvalue problem with a single positive-valued design variable ϵ. First-order Taylor series expansion can then be applied to estimate the change in the repeated eigenvalues and their corresponding eigenvectors caused by the changes in design variables. In this technique the multivariable structural eigenvalue problems defined in the neighborhood of the repeated eigenvalues are reparameterized in terms of the single parameter, ϵ, ranging from 0 to 1. Consequently, the design space becomes one dimensional and the changes in designs are always positive. Therefore, the reordering of the eigenvectors is solely dependent on the magnitudes of the eigenvalue derivatives. This technique has been validated by analytical as well as numerical examples.

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